18.06 MIDTERM 1

(Note: those of you starting at 11+x hours, please add x hours to all the times below)

- *10:55 - exam available on Canvas (email jhahn01@mit.edu ASAP if you can't access it)*
- *11:05 - you may start reading the problems and working on them*
- *11:55 - you must stop writing at this time*
- *12:10 - Gradescope upload cut-off. If you can't upload your finished exam to Gradescope for technical reasons, please email it to jhahn01@mit.edu before this time*

NO collaboration or written/electronic/online sources allowed, except for our course materials (lecture and recitation notes, problem sets and review session problems + solutions).

DOWNLOAD or **DOWNLOAD + PRINT** this exam before the official start time. You can either annotate the PDF file, or physically write on paper (if you need extra sheets of paper, please write your name and the problem number on **ALL** pages that you want graded).

UPLOAD or **SCAN + UPLOAD** your exam to Gradescope after time is up and pencils are down. Make sure that the pages are easily legible (e.g. good camera quality and angle).

You **MUST** show your work to receive credit. **JUSTIFY EVERYTHING**. Just giving a correct answer without an explanation of what led you to it will be **SEVERELY** penalized.

There are 3 problems, each with 4 parts.

NAME: ______________________________

MIT ID NUMBER: ______________________________

RECITATION INSTRUCTOR: ______________________________

PROBLEM 1 **NAME:** ____________________

(1) Use Gaussian elimination to put the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ in row echelon form.

Show all your steps! *(10 pts)*

(2) Use part (1) to write $A = LU$, where L is lower triangular and U is upper triangular.

Then express L as a product of elimination matrices $E_{ij}^{(\lambda)}$ for various $i > j$ and numbers λ.

(10 pts)

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(3) Find a linear combination of the columns of A which is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *Hint: the answer to (1) may help.* *(5 pts)*

(4) Explain why for any 4×4 matrix X, the product AX cannot be invertible. *(5 pts)*

PROBLEM 2 **NAME:** ____________________

Consider the system of equations:

$$\begin{cases} \quad a - 2b + \;\; 6c = 1 \\ -2a + 3b - 11c = -3 \end{cases} \qquad (*)$$

(1) Write the system as $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{b}$ for a suitably chosen 2×3 matrix A and 2×1 vector $\mathbf{b}$.

(5 pts)

(2) Use Gauss-Jordan elimination to put A from part (1) in **reduced** row echelon form.

Show all your steps! *Hint: we recommend you actually do Gauss-Jordan elimination on the extended matrix* $\left[\ A \mid \mathbf{b}\ \right]$*; it's a little bit more work, but it will pay off in part (4).*

(10 pts)

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(3) Write down the vector(s) in a basis for the nullspace of A. What is the dimension of this nullspace? **Explain how you know!** *(10 pts)*

(4) What is the general (i.e. complete) solution of the system $(*)$? *(10 pts)*

PROBLEM 3 **NAME:** ____________________

(1) Let V be the following vector subspace of $\mathbb{R}^2$:

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } 3x + 4y = 0 \right\}$$

Find a basis for $W = V^\perp$ (in other words, W is the orthogonal complement of V). *(5 pts)*

In what follows, you may use the formula $P_{C(A)} = A(A^T A)^{-1} A^T$ for the projection matrix onto the column space $C(A)$ of any matrix A

(2) Compute the projection matrices P_V and P_W onto the subspaces from part (1). *(10 pts)*

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(3) Compute $P_V P_W$ and $P_W P_V$, where P_V and P_W are as in part (2). *(10 pts)*

(4) Based on part (3), formulate a general principle by filling the blanks below:

> *For any vector spaces V and W, the projection matrices have the property that*
>
> *$P_V P_W$ and $P_W P_V$ are* ____________ *if V and W are* ______________________

After formulating the principle above, justify it using a geometric argument (i.e. using the geometric interpretation of projections). *(10 pts)*